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Characterizations of Hartley Proper Efficiency in Nonconvex Vector Optimization

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Abstract. This paper deals with characterizations of Hartley proper efficiency in a vector optimization problem involving nonconvex and nondifferentiable functions. The case when objective and constraint functions are locally Lipschitz is also considered. Sufficient conditions for Hartley proper efficiency in locally Lipschitz programs are given under a near invex-infineness assumption first introduced in this paper.

Key words: Invexity, Nonconvex program, Proper efficiency, Vector Optimization, Infineness

1. Introduction

Since the appearance of the paper [15] where a notion of proper efficiency was first introduced, several authors [2–6, 9, 11, 13] have proposed various modified versions of this notion. A comprehensive survey of proper efficiencies can be found in [10]. Characterizations of these notions are mainly obtained in problems with generalized convexity structure (see [2–6, 9, 11, 13]). The case of set-valued maps is considered in [16–18, 21]. Benson and Morin [1] gave an important result characterizing Geoffrion proper efficiency for a convex vector optimization problem in terms of the stability for a related scalar optimization problem. Recently, Huang and Yang [14] studied a general vector optimization problem without any convexity assumption and obtained two characterizations of Geoffrion proper efficiency one of which (see [14, Theorem 3.2]) was given by means of a stability property of a related scalar optimization problem. The other characterization of [14, Theorem 3.1] is formulated in terms of the existence of an exact penalty function of a constrained program.

In this paper we prove several characterizations of Hartley proper efficiency [11] for a vector optimization problem where no convexity structure is required to be satisfied, and the cone defining the partial order of Euclidean spaces is an arbitrary closed convex pointed cone. A detailed discussion is devoted to the case when this cone is polyhedral. Our characterizations given in Theorems 2.1, 2.2, 2.4–2.6 show that problems of finding Hartley properly efficient points are equivalent to some scalar optimization problems. A modified version of [14, Theorem 3.2] (see our Theorem 2.3) is introduced to prove that, similarly to the Geoffrion proper efficiency property, the Hartley proper efficiency can be characterized in terms of the stability of suitable scalar optimization problems. We also consider problems when objective and constraint functions are locally Lipschitz. Necessary conditions for Hartley proper efficiency in nonsmooth vector optimization problems are obtained by combining one of our characterizations with scalar optimization results of Clarke [7]. These necessary conditions become sufficient conditions for Hartley proper efficiency if inequality constraint functions are nearly invex and equality constraint functions are nearly infine. Here our notions of near invexity and near infineness are more general than the corresponding notions of invexity and infineness introduced in [19]. We also give several characterizations of near invexity and near infineness, and provide one example proving that the class of nearly invex (resp. nearly infine) maps is strictly broader than the class of invex (resp. infine) maps.

2. Reduction Theorems: Characterizations of Hartley Proper Efficiency

In this paper each element of a s-dimensional Euclidean space $\mathbb{R}^s$ is identified with a column-vector i.e., an $s \times 1$-matrix. So the inner product of two vectors a and b of $\mathbb{R}^s$ can be written as $a^\tau b$ where τ denotes the transpose. We denote by $\mathbb{R}^s_+$ the nonnegative orthant of $\mathbb{R}^s$. We write $\mathbb{R}$ instead of $\mathbb{R}^1$, and $\mathbb{R}_+$ instead of $\mathbb{R}^1_+$. We use the symbols $\overline{A}$, co$A$ and int$A$ to denote the closure, the convex hull and the interior of $A \subset \mathbb{R}^s$. The empty set is denoted by $\emptyset$.

In this section we show that problems of finding a properly efficient point of vector optimization problems can be reduced to problems of scalar optimization. This suggests us to use the terminology ''reduction theorems'' in this section. Proper efficiency in this paper is understood in the sense of Hartley [11]. We first give the exact definition of this notion.

Let $f: \mathbb{R}^n \to \mathbb{R}^s$ be an arbitrary vector-valued map with components f^i, $i = 1, 2, \ldots, s$, and Q be a nonempty subset of $\mathbb{R}^n$. Let $D \subset \mathbb{R}^s$ be a closed convex pointed cone. Recall that D is pointed if $y \in D \cap -D \Longrightarrow y = 0$.

Consider the following vector optimization problem (P):

$$\text{minimize} \quad f(x) := (f^1(x), f^2(x), \ldots, f^s(x))^\tau$$
$$\text{subject to} \quad x \in Q.$$

A point $x_0 \in Q$ is called a D-efficient point of (P) if

$$\forall x \in Q, \qquad f(x) - f(x_0) \notin -D \setminus \{0\}.$$

A point $x_0 \in Q$ is called a Hartley properly efficient point (or more exactly, a Hartley properly D-efficient point) of (P) if it is a D-efficient point of (P) and if there exists a positive number M such that for each $\zeta \in D^+ \cap S$ and $x \in Q$ with $\zeta^\tau(f(x) - f(x_0)) < 0$ there exists $\tilde{\zeta} \in D^+ \cap S$ such that

$$\tilde{\zeta}^\tau(f(x) - f(x_0)) > 0 \tag{2.1}$$

and

$$\frac{\zeta^\tau(f(x_0) - f(x))}{\tilde{\zeta}^\tau(f(x)) - f(x_0))} \leqslant M \tag{2.2}$$

where

$$D^+ = \{y \in \mathbb{R}^s : d^\tau y \geqslant 0 \ \forall d \in D\},$$
$$S = \{y \in \mathbb{R}^s : \|y\| = 1\}.$$

Observe that $D^+ \cap S \neq \emptyset$ by Proposition 2.1.4 of [20]. A number $M > 0$ satisfying the requirement of the above definition of Hartley proper efficiency of x_0 is called a Hartley constant of (P) at x_0. The set of all Hartley constants of (P) at x_0 is denoted by $H(x_0)$.

Let us fix a point $x_0 \in Q$ and consider the following function $\bar{f}$ depending on parameters $M > 0$ and $\zeta \in D^+$:

$$\bar{f}(x) = \bar{f}(M, \zeta, x) = \zeta^\tau(f(x) - f(x_0)) + M\|\zeta\|\rho(0, f(x) - f(x_0) + D), \tag{2.3}$$

where $\rho(y, A)$ denotes the distance from $y \in \mathbb{R}^s$ to $A \subset \mathbb{R}^s$. We observe that the subset

$$D^{+i} := \{y \in \mathbb{R}^s : d^\tau y > 0 \ \ \forall d \in D \setminus \{0\}\}$$

of D^+ is nonempty. We now give the first characterization of Hartley proper efficiency.

THEOREM 2.1. *Let $x_0 \in Q$. If x_0 is a Hartley properly efficient point of* (P) *and $M \in H(x_0)$ then for any $\zeta \in D^+$*

$$\min_{x \in Q} \bar{f}(x) = \bar{f}(x_0) = 0, \tag{2.4}$$

where $\bar{f}$ is defined by (2.3). *Conversely, if there exist $M > 0$ and $\zeta \in D^{+i}$ such that function* (2.3) *satisfies condition* (2.4) *then x_0 is a Hartley properly efficient point of* (P).

Proof. We first observe that for any $y \in \mathbb{R}^s$

$$\inf_{d\in D} \zeta^\tau(y+d) = \begin{cases} \zeta^\tau y & \text{if} \quad \zeta \in D^+ \\ -\infty & \text{if} \quad \zeta \notin D^+. \end{cases}$$

Making use of this observation and a minimax theorem we see that

$$\begin{aligned} \rho(0, f(x) - f(x_0) + D) &= \inf_{d\in D} \|f(x) - f(x_0) + d\| \\ &= \inf_{d\in D} \max_{\zeta\in B} \zeta^\tau(f(x) - f(x_0) + d) \\ &= \max_{\zeta\in B} \inf_{d\in D} \zeta^\tau(f(x) - f(x_0) + d) \\ &= \max_{\zeta\in D^+\cap B} \zeta^\tau(f(x) - f(x_0)), \end{aligned} \tag{2.5}$$

where B denotes the closed unit ball of $\mathbb{R}^s$.

To prove the necessity part of Theorem 2.1 it suffices to show that $\inf_{x\in Q} \bar{f}(x) \geqslant 0$ (since $\bar{f}(x_0) = 0$). Indeed, if $\zeta = 0$ then this inequality is clear. Consider now the case $\zeta \in D^+ \setminus \{0\}$. Assume to the contrary that $\bar{f}(x) < 0$ for some $x \in Q$. Then, setting $\bar{\zeta} = \|\zeta\|^{-1}\zeta$ we derive from (2.3) and (2.5) that

$$\begin{aligned} &\bar{\zeta}^\tau(f(x) - f(x_0)) < 0, \\ &\bar{\zeta}^\tau(f(x) - f(x_0)) + M\hat{\zeta}^\tau(f(x) - f(x_0)) < 0 \end{aligned}$$

for all $\hat{\zeta} \in D^+ \cap B$. This contradicts the Hartley proper efficiency of x_0.

To prove the sufficiency part of Theorem 2.1 we observe from (2.3) to (2.5) that

$$\max_{\hat{\zeta}\in D^+\cap B} (\zeta + M\|\zeta\|\hat{\zeta})^\tau(f(x) - f(x_0)) \geqslant 0, \quad x \in Q.$$

This implies that

$$\max_{\hat{\zeta}\in D^+\cap B} (\zeta + M\|\zeta\|\hat{\zeta})^\tau(f(x) - f(x_0) + d) \geqslant 0, \quad \forall x \in Q, \quad \forall d \in D, \tag{2.6}$$

since $\zeta + M\|\zeta\|\hat{\zeta} \in D^+$ for all $\hat{\zeta} \in D^+ \cap B$. From (2.6) and the continuity property we get

$$\max_{\hat{\zeta}\in D^+\cap B} (\zeta + M\|\zeta\|\hat{\zeta})^\tau y \geqslant 0 \tag{2.7}$$

for all $y \in \overline{\text{cone}}(f(Q) - f(x_0) + D)$ where cone $A = \{\lambda a : \lambda > 0, \ a \in A\}$ and $\overline{\text{cone}}\, A = \overline{\text{cone}\, A}$. On the other hand, for all $d \in -D \setminus \{0\}$

$$\max_{\hat{\zeta}\in D^+\cap B} (\zeta + M\|\zeta\|\hat{\zeta})^\tau d < 0, \tag{2.8}$$

since $\zeta + M\|\zeta\|\hat{\zeta} \in D^{+i}$ for all $\hat{\zeta} \in D^+ \cap B$. Combining (2.7) and (2.8) yields

$$-D \cap \overline{\text{cone}}(f(Q) - f(x_0) + D) = \{0\}.$$

Observe that the last condition is exactly the definition of Benson proper efficiency of x_0 (see [2]). Thus, if there exist $M > 0$ and $\zeta \in D^{+i}$ such that (2.4) holds then x_0 is a Benson properly efficient point of (P). To conclude the proof it remains to note that under our assumptions imposed on D the notions of Benson proper efficiency and Hartley proper efficiency coincide (see [10, p. 9]). □

Before formulating Theorem 2.2 let us introduce the following function depending on a parameter $M > 0$:

$$\bar{F}(x) = \bar{F}(M, x) = t(x, x_0) + M\rho(0, f(x) - f(x_0) + D), \quad x \in Q, \tag{2.9}$$

where

$$t(x, x_0) = \min_{\zeta \in D^+ \cap S} \zeta^\tau (f(x) - f(x_0)). \tag{2.10}$$

THEOREM 2.2.

1. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *if and only if there exist* $M > 0$ *and* $\zeta \in D^{+i}$ *such that function* (2.3) *satisfies condition* (2.4).
2. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *if and only if there exists* $M > 0$ *such that*

$$\min_{x \in Q} \bar{F}(x) = \bar{F}(x_0) = 0, \tag{2.11}$$

where $\bar{F}$ *is defined by* (2.9).

Proof. Observe that $D^{+i} \neq \emptyset$ since D is a closed convex pointed cone. Therefore the first part of Theorem 2.2 is immediate from Theorem 2.1. Let us prove the second one. If x_0 is a Hartley properly efficient point of (P) then by Theorem 2.1 there exists $M > 0$ such that

$$\min_{\zeta \in D^+ \cap S} \min_{x \in Q} \bar{f}(M, \zeta, x) = \min_{\zeta \in D^+ \cap S} \bar{f}(M, \zeta, x_0) = 0.$$

To obtain (2.11) it remains to observe that

$$\begin{aligned} \min_{\zeta \in D^+ \cap S} \min_{x \in Q} \bar{f}(M, \zeta, x) &= \min_{x \in Q} \min_{\zeta \in D^+ \cap S} \bar{f}(M, \zeta, x) \\ &= \min_{x \in Q} \bar{F}(M, x). \end{aligned}$$

Conversely, let (2.11) hold where $\bar{F}$ is defined by (2.9). Let ζ be an arbitrary point of D^{+i}. Then

$$t(x, x_0) \leqslant \|\zeta\|^{-1} \zeta^\tau (f(x) - f(x_0)), \quad x \in Q.$$

Hence (2.11) $\Rightarrow$ (2.4) where $\bar{f}(x) = \bar{f}(M, \|\zeta\|^{-1}\zeta, x)$. The Hartley proper efficiency of x_0 is thus derived from Theorem 2.1. □

COROLLARY 2.1. *Let x_0 be a D-efficient point of* (P) *and f be not constant on Q. Then $Q(x_0) = \{x \in Q : f(x_0) - f(x) \notin D\} \neq \emptyset$, and x_0 is a Hartley properly efficient point of* (P) *if and only if*

$$q(x_0) := \sup_{x \in Q(x_0)} \frac{-t(x, x_0)}{\rho(0, f(x) - f(x_0) + D)} < +\infty. \tag{2.12}$$

Proof. Observe that x_0 is a D-efficient point of (P) if and only if $x \in Q \setminus Q(x_0) \Rightarrow f(x_0) = f(x)$. This proves that $Q(x_0) \neq \emptyset$ if f is not constant on Q. Also, if x_0 is a D-efficient point of (P) then

$$\begin{aligned} & x \in Q \setminus Q(x_0) \text{ (i.e. } \rho(0, f(x) - f(x_0) + D) = 0) \\ & \quad \Rightarrow f(x) = f(x_0) \\ & \quad \Rightarrow t(x, x_0) = 0. \end{aligned}$$

This shows that if x_0 is a D-efficient point of (P) then $[\bar{F}(M, x) \geqslant 0, \quad \forall x \in Q] \Leftrightarrow [\bar{F}(M, x) \geqslant 0, \quad \forall x \in Q(x_0)]$. From this and from the second part of Theorem 2.2 we obtain Corollary 2.1. □

Before pointing out an estimate for the infimum of the set $H(x_0)$ let us introduce the following definition: a point $x_0 \in Q$ is called an ideal point of (P) if for all $x \in Q$, $f(x) - f(x_0) \in D$ i.e., $t(x, x_0) \geqslant 0$. If D is the non-negative orthant of $\mathbb{R}^s$ this property means that $f^i(x) \geqslant f^i(x_0)$ for all $i = 1, 2, \ldots, s$, and $x \in Q$. Such a property is rarely seen in practice. Obviously, an ideal point is a Hartley properly efficient point, but the converse statement is no longer true.

COROLLARY 2.2. *Let $x_0 \in Q$ be a Hartley properly efficient point of* (P). *Then*

1. $0 = \inf\{M : M \in H(x_0)\}$ *if x_0 is an ideal point of* (P).
2. $0 < q(x_0) \leq \inf\{M : M \in H(x_0)\}$ *if x_0 is not an ideal point of* (P).

Proof. The first assertion of Corollary 2.2 is clear. Indeed, if x_0 is an ideal point of (P) then, for all $x \in Q$ and $\zeta \in D^+ \cap S$, $\zeta^\tau(f(x) - f(x_0)) \geqslant 0$ and hence, any positive number M can be taken as a Hartley constant at x_0. This means that $0 = \inf\{M{:}M \in H(x_0)\}$.

To prove the second assertion of Corollary 2.2 let us assume that x_0 is not an ideal point of (P). We have seen in the proof of Corollary 2.1 that

$$x \in Q \setminus Q(x_0) \Rightarrow t(x, x_0) = 0.$$

So, if $Q(x_0) = \emptyset$ or if $t(x, x_0) \geqslant 0$ for all $x \in Q(x_0)$ then x_0 must be an ideal point of (P). From this remark and from the assumption that x_0 is not an ideal point of (P) we obtain $Q(x_0) \neq \emptyset$ and $t(x, x_0) < 0$ for some $x \in Q(x_0)$. This proves that $q(x_0) > 0$ (see 2.12)). In addition, we have seen in the proof of Corollary 2.1 that $M \in H(x_0) \Rightarrow M \geqslant q(x_0)$. In other words, $\inf\{M{:}M \in H(x_0)\} \geqslant q(x_0)$. □

REMARK 2.1. From Corollary 2.2 we see that if $x_0 \in Q$ is a Hartley properly efficient point of (P) then

$$0 = \inf\{M : M \in H(x_0)\} \iff x_0 \text{ is an ideal point of (P).}$$
$$0 < \inf\{M : M \in H(x_0)\} \iff x_0 \text{ is not an ideal point of (P).}$$

COROLLARY 2.3. *If there exists $\zeta \in D^{+i}$ such that the function $\zeta^{\tau} f(\cdot)$ attains its minimum on Q* at *$x_0 \in Q$ then x_0 is a Hartley properly efficient point of* (P).

Proof. Apply Theorem 2.2 and note that the assumptions of Corollary 2.3 imply (2.4). □

REMARK 2.2. Corollary 2.3 was established in [11, Theorem 6.2]. Before giving other characterizations of Hartley proper efficiency let us consider the following scalar optimization problem $(\bar{P}_0)$:

$$\begin{array}{ll} \text{minimize} & \varphi(x) \\ \text{subject to} & x \in Q, \\ & f(x) \leqslant_D 0, \end{array}$$

where $\varphi{:}\mathbb{R}^n \to \mathbb{R}$ is an arbitrary function and $y_1 \leqslant_D y_2$ means that $y_2 - y_1 \in D$.

For each $y \in \mathbb{R}^s$ we associate to $(\bar{P}_0)$ a perturbed minimization problem, denoted by $(\bar{P}_y)$:

$$\begin{array}{ll} \text{minimize} & \varphi(x) \\ \text{subject to} & x \in Q, \\ & f(x) \leqslant_D y. \end{array}$$

Let us denote by $A(y)$ the set of all points x satisfying the constraints of Problem $(\bar{P}_y)$. We set

$$v(y) = \begin{cases} \inf\{\varphi(x) : x \in A(y)\} & \text{if } A(y) \neq \emptyset, \\ +\infty & \text{if } A(y) = \emptyset. \end{cases}$$

Let x_0 be a minimizer of $(\bar{P}_0)$ i.e.

$$v(0) = \inf\{\varphi(x) : x \in A(0)\} = \varphi(x_0).$$

We say that Problem $(\bar{P}_0)$ is stable at x_0 if there exists $M > 0$ such that

$$\frac{v(y) - v(0)}{\|y\|} \geqslant -M, \quad y \neq 0. \tag{2.13}$$

Stability property of $(\bar{P}_0)$ is characterized by the following lemma.

LEMMA 2.1. *Let x_0 be a minimizer of $(\bar{P}_0)$. Then Problem $(\bar{P}_0)$ is stable at x_0 if and only if there exists $M > 0$ such that x_0 is a minimizer of the function*

$$p(\cdot) := \varphi(\cdot) + M\rho(0, f(\cdot) + D)$$

on the set Q.

Proof. (i) *Necessity*. Assume that $(\bar{P}_0)$ is stable, but x_0 is not a minimizer of $p(\cdot)$ on Q. Then for some $x \in Q$

$$\varphi(x) < \varphi(x_0) - M\rho(0, f(x) + D). \tag{2.14}$$

Observe that $\rho(0, f(x) + D) > 0$ since x_0 is a minimizer of $(\bar{P}_0)$. Let $y \in f(x) + D$ be such that $\|y\| = \rho(0, f(x) + D)$. Then $\|y\| > 0$, $x \in A(y)$ and by (2.14)

$$v(y) \leqslant \varphi(x) < v(0) - M\|y\|,$$

a contradiction to (2.13).

(ii) *Sufficiency*. By assumption

$$\varphi(x_0) \leqslant \varphi(x) + M\rho(0, f(x) + D), \quad x \in Q.$$

Hence for all $y \in \mathbb{R}^s \setminus \{0\}$ and $x \in A(y)$

$$\varphi(x_0) \leqslant \varphi(x) + M\|y\|,$$

which implies that

$$v(0) \leqslant \inf\{\varphi(x) : x \in A(y)\} + M\|y\| = v(y) + M\|y\|$$

i.e., (2.13) holds, as required. □

As a consequence of Theorem 2.2 and Lemma 2.1 we obtain the following theorem.

THEOREM 2.3. *Let* $x_0 \in Q$ *be a D-efficient point of* (P). *Then*

1. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *if and only if there exists* $\zeta \in D^{+i}$ *such that the following Problem* (P_ζ) *is stable at* x_0:

$$\begin{aligned} &\text{minimize } \zeta^\tau f(x) \\ &\text{subject to } x \in Q, \\ &\qquad f(x) - f(x_0) \leqslant_D 0. \end{aligned}$$

2. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *if and only if the following problem is stable at* x_0:

$$\begin{aligned} &\text{minimize } t(x, x_0) \\ &\text{subject to } x \in Q, \\ &\qquad f(x) - f(x_0) \leqslant_D 0. \end{aligned}$$

Let us observe that x_0 is a minimizer of each of the scalar optimization problems mentioned in Theorem 2.3. This is a consequence of the assumption that x_0 is a D-efficient point of (P).

REMARK 2.3. Characterization of Geoffrion proper efficiency in terms of stability property can be found in [14, Theorem 3.2]. Our Theorem 2.3 is a modified version of [14, Theorem 3.2] for the case of Hartley proper efficiency.

From now on we assume that D is a polyhedral cone given by

$$D = \{y \in \mathbb{R}^s : d_i^\tau y \geqslant 0, \ i \in I\}, \tag{2.15}$$

where $I = \{1, 2, \ldots, m\}$ and d_i are fixed points of $\mathbb{R}^s$ with $\|d_i\| = 1$. We also assume that D is pointed. Observe that if D is defined by (2.15) then

$$D^+ = \left\{ \sum_{i \in I} \alpha^i d_i \ : \ \alpha^i \geqslant 0, \ i \in I \right\}. \tag{2.16}$$

It is known from [11, Theorem 6.1] that x_0 is a Hartley properly efficient point of (P) if and only if x_0 is a D-efficient point of (P) and there exists a number $M > 0$ such that for each $x \in Q$ and $i \in I$ with $d_i^\tau(f(x) - f(x_0)) < 0$ there exists $j \in I$ such that

$$d_j^\tau(f(x) - f(x_0)) > 0 \tag{2.17}$$

and

$$\frac{d_i^\tau(f(x_0) - f(x))}{d_j^\tau(f(x) - f(x_0))} \leqslant M. \tag{2.18}$$

If $d_i, i \in I$, are the ith unit vectors of $\mathbb{R}^s$(i.e.,$d_i = (0, \ldots, 0, 1, 0, \ldots, 0)^\tau \in \mathbb{R}^s$ where the ith component of d_i is 1 and other components of d_i is 0) and if $s = m$ then $D = \mathbb{R}^s_+$ and the Hartley properly efficient point coincides with the Geoffrion properly efficient solution defined in [9].

THEOREM 2.4. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *and* $M \in H(x_0)$ *if and only if for each* $i \in I$

$$\min_{x \in Q} \bar{f}^i(x) = \bar{f}^i(x_0) = 0, \tag{2.19}$$

where

$$\bar{f}^i(x) = \max_{j \in I}(d_i + M d_j)^\tau (f(x) - f(x_0)). \tag{2.20}$$

Proof. (i) *Necessity*. It is easy to see that for each $i \in I$, $\inf_{x \in Q} \bar{f}^i(x) \geqslant 0$. Indeed, otherwise there exists $i \in I$ such that

$$\inf_{x \in Q} \bar{f}^i(x) < 0,$$

and hence there exists $x \in Q$ such that $\bar{f}^i(x) < 0$, that is,

$$\text{for all } j \in I, \quad (d_i + Md_j)^\tau (f(x) - f(x_0)) < 0. \tag{2.21}$$

From (2.21), we have for $j = i$

$$(1 + M)d_i^\tau (f(x) - f(x_0)) < 0.$$

Hence $d_i^\tau (f(x) - f(x_0)) < 0$. Thus by the definition of Hartely proper efficiency we must find $j \in I \setminus \{i\}$ such that (2.17) and (2.18) hold. This implies that

$$(d_i + Md_j)^\tau (f(x) - f(x_0)) \geqslant 0,$$

which contradicts (2.21). We have thus proved that

$$\text{for each } i \in I, \quad \inf_{x \in Q} \bar{f}^i(x) \geqslant 0.$$

But $\bar{f}^i(x_0) = 0$. Thus for each $i \in I$,

$$\min_{x \in Q} \bar{f}^i(x) = \bar{f}^i(x_0) = 0.$$

(ii) *Sufficiency*. We first claim that x_0 is a D-efficient point. Indeed, otherwise there exists $x \in Q$ such that

$$f(x) - f(x_0) \in -D \setminus \{0\}. \tag{2.22}$$

From (2.22),

$$d_i^\tau (f(x) - f(x_0)) \leqslant 0 \quad \text{for all } j \in I \tag{2.23}$$

and

$$d_i^\tau (f(x) - f(x_0)) < 0 \quad \text{for some } i \in I. \tag{2.24}$$

Indeed, it is clear that (2.23) holds. Now, if for all $i \in I$, $d_i^\tau (f(x) - f(x_0)) = 0$, then $(f(x) - f(x_0)) \in D$, but since D is pointed, $(f(x) - f(x_0)) \in D \cap (-D) = \{0\}$, and hence $f(x) = f(x_0)$, which contradicts (2.22). Thus (2.24) holds. From (2.23) to (2.24), for each $j \in I$,

$$(d_i + Md_j)^\tau (f(x) - f(x_0)) < 0, \text{ i.e., } \bar{f}^i(x) < 0,$$

which contradicts (2.19).

To prove that x_0 is a Hartley properly efficient point of (P), let us take $x \in Q$ and $i \in I$ with $d_i^\tau (f(x) - f(x_0)) < 0$. Since by assumption $\bar{f}^i(x) \geqslant 0$, we must find an index j such that

$$(d_i + Md_j)^\tau (f(x) - f(x_0)) \geqslant 0.$$

This proves that M satisfies the requirement of the Hartley proper efficiency of x_0. □

COROLLARY 2.4. *Let* x_0 be a *D-efficient point of* (P) *and f be not constant on* Q. Then

$$Q'(x_0) := \left\{ x \in Q : \max_{j \in I} d_j^{\tau}(f(x) - f(x_0)) > 0 \right\} \neq \emptyset$$

and x_0 *is a Hartley properly efficient point of* (P) *if and only if*

$$q'(x_0) := \sup_{x \in Q'(x_0)} \frac{-\min_{j \in I} d_j^{\tau}(f(x) - f(x_0))}{\max_{j \in I} d_j^{\tau}(f(x) - f(x_0))} < +\infty.$$

Proof. Observe that x_0 is a D-efficient point of (P) if and only if for all $x \in Q$

$$\max_{j \in I} d_j^{\tau}(f(x) - f(x_0)) \leqslant 0 \;\Rightarrow\; f(x) = f(x_0).$$

Since f is not constant on Q it follows from this observation that $Q'(x_0) \neq \emptyset$. Also, if x_0 is a D-efficient point of (P) then $x \in Q \setminus Q'(x_0) \;\Rightarrow\; f(x) = f(x_0)$. Hence

$$[\bar{f}^i(x) \geqslant 0, \quad \forall x \in Q] \Leftrightarrow [\bar{f}^i(x) \geqslant 0, \quad \forall x \in Q'(x_0)].$$

Combining this with Theorem 2.4 we see that x_0 is a Hartley properly efficient point of (P) and $M \in H(x_0)$ if and only if, for all $i \in I$ and $x \in Q'(x_0)$,

$$\frac{d_i^{\tau}(f(x_0) - f(x))}{\max_{j \in I} d_j^{\tau}(f(x) - f(x_0))} \leqslant M.$$

This is equivalent to saying that x_0 is a Hartley properly efficient point of (P) and $M \in H(x_0)$ if and only if $q'(x_0) \leqslant M$. Corollary 2.4 is thus established. □

COROLLARY 2.5. *Let* $x_0 \in Q$ *be a Hartley properly efficient point of* (P). *Then*

1. $0 = \inf\{M : M \in H(x_0)\}$ *if* x_0 *is an ideal point of* (P).
2. $0 < q'(x_0) \leqslant \inf\{M : M \in H(x_0)\}$ *if* x_0 *is not an ideal point of* (P).

Proof. The first assertion of Corollary 2.5 is clear. To prove the second one let us assume that x_0 is not an ideal point of (P). Since x_0 is a D-efficient point of (P) we have seen in the proof of Corollary 2.4 that

$$x \in Q \setminus Q'(x_0) \Rightarrow f(x) - f(x_0) = 0.$$

Thus $Q'(x_0) \neq \emptyset$. Indeed, otherwise for all $x \in Q$ $f(x) - f(x_0) = 0 \in D$, a contradiction to the assumption that x_0 is not an ideal point. To prove that $0 < q'(x_0)$ it suffices to show that, for some $x \in Q'(x_0)$, $\min_{j \in I} d_j^\tau(f(x) - f(x_0)) < 0$. Indeed, otherwise $\min_{j \in I} d_j^\tau(f(x) - f(x_0)) \geq 0$ (i.e., $f(x) - f(x_0) \in D$) for all $x \in Q'(x_0)$. On the other hand, $f(x) - f(x_0) = 0 \in D$ for all $x \in Q \setminus Q'(x_0)$. This proves that for all $x \in Q$ $f(x) - f(x_0) \in D$, a contradiction to the assumption that x_0 is not an ideal point of (P). Thus, inequality $0 < q'(x_0)$ is established. To see that $q'(x_0) \leqslant \inf\{M : M \in H(x_0)\}$ it is enough to remark from the proof of Corollary 2.4 that $M \in H(x_0) \Rightarrow q'(x_0) \leqslant M$.

Before going further let us consider the following lemma.

LEMMA 2.2. *Let* $M > 0$, $b = (b^1, b^2, \ldots, b^m) \in \mathbb{R}^m$ *and* $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m_+ \setminus \{0\}$. *Then* $c \geqslant 0 \Leftrightarrow c' \geqslant 0$ *where*

$$c = \sum_{i \in I} \alpha^i b^i + M \max(b^1, b^2, \ldots, b^m),$$

$$c' = \sum_{i \in I} \alpha^i b^i + M \ \max(b^1, b^2, \ldots, b^m, 0).$$

Proof. Let $\beta = \max(b^1, b^2, \ldots, b^m)$ and $\beta' = \max(b^1, b^2, \ldots, b^m, 0)$. Since $\beta' \geq \beta$ we obviously have implication $c \geqslant 0 \Rightarrow c' \geqslant 0$. To complete our proof it suffices to show that $c < 0 \Rightarrow c' < 0$. Indeed, if $\beta < 0$ then $b^i < 0$ for all $i \in I$, and $\beta' = 0$. Therefore $c' = \sum_{i \in I} \alpha^i b^i < 0$. If $\beta \geq 0$ then $\beta = \beta'$ and hence $c = c'$. This proves that $c < 0 \Rightarrow c' < 0$, as required. □

COROLLARY 2.6. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *and* $M \in H(x_0)$ *if and only if for each* $i \in I$

$$\min_{x \in Q} \bar{F}^i(x) = \bar{F}^i(x_0) = 0.$$

where

$$\bar{F}^i(x) = d_i^\tau(f(x) - f(x_0)) + M \max(d_1^\tau(f(x) - f(x_0)), \ldots, d_m^\tau(f(x) - f(x_0)), 0).$$

Proof. For fixed $i \in I$ let us set $\alpha^i = 1$ and $\alpha^j = 0$, $j \neq i$. Let $b^j = d_j^\tau(f(x) - f(x_0))$, $j \in I$. Then for each $x \in Q$ by Lemma 2.2 $\bar{f}^i(x) \geq 0 \Leftrightarrow \bar{F}^i(x) \geq 0$. Thus x_0 is a minimizer of $\bar{f}^i$ on Q if and only if x_0 is a minimizer of $\bar{F}^i$ on Q. From this remark and Theorem 2.4 we obtain Corollary 2.6. □

Now for $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m_+$ and $M > 0$ let us introduce the following function

$$\begin{aligned} \hat{f}(x) = \hat{f}(M, \alpha, x) &= \sum_{i\in I} \alpha^i d_i^\tau (f(x) - f(x_0)) \\ &\quad + M\left(\sum_{i\in I} \alpha^i\right) \max_{j\in I} d_j^\tau (f(x) - f(x_0)), \quad x \in Q. \end{aligned} \tag{2.25}$$

THEOREM 2.5. *Let* $x_0 \in Q$. *If* x_0 *is a Hartley properly efficient point of* (P) *and* $M \in H(x_0)$ *then for all* $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m_+$ *the function* (2.25) *satisfies the following condition*

$$\min_{x\in Q} \hat{f}(x) = \hat{f}(x_0) = 0. \tag{2.26}$$

Conversely, if there exist $M > 0$ *and a vector* $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m_+$ *with positive components* $\alpha^i, i \in I$, *such that the function* (2.25) *satisfies condition* (2.26) *then* x_0 *is a Hartely properly efficient point of* (P).

Proof.

(i) *Necessity*. This is a consequence of Theorem 2.4. Indeed, multiplying equalities (2.19) by α^i and summing up the obtained equalities we get (2.26), as desired.

(ii) *Sufficiency*. Since D^+ is defined by (2.16) and since D is a pointed cone we can see that $\sum_{j\in I} \alpha^j d_j \in D^{+i}$ for all $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m_+$ with $\alpha^j > 0,\ j \in I$. Now setting $\beta = \sum_{i\in I} \alpha^i$ and $\zeta = \beta^{-1} \sum_{j\in I} \alpha^j d_j \in D^{+i}$ we derive from (2.26) that

$$\min_{x\in Q} \tilde{f}(x) = \tilde{f}(x_0) = 0, \tag{2.27}$$

where

$$\tilde{f}(x) := \zeta^\tau (f(x) - f(x_0)) + M \max_{j\in I} d_j^\tau (f(x) - f(x_0)), \quad x \in Q.$$

Observing that $d_j \in D^+ \cap S,\ j \in I$, and taking account of (2.5) we see that

$$\bar{f}(M, \zeta, x) \geq \tilde{f}(x), \quad x \in Q,$$

and $\bar{f}(M, \zeta, x_0) = \tilde{f}(x_0) = 0$. From this and (2.27) we claim that (2.4) holds. To complete our proof it remains to apply Theorem 2.1 □

COROLLARY 2.7. *If* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *and* $M \in H(x_0)$ *then for all* $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m_+$

$$\min_{x\in Q} \hat{F}(x) = \hat{F}(x_0) = 0, \tag{2.28}$$

where

$$\hat{F}(x) = \hat{F}(M, \alpha, x) = \sum_{i\in I} \alpha^i d_i^\tau (f(x) - f(x_0)) + M\left(\sum_{i\in I}\alpha^i\right) \max(d_1^\tau(f(x) - f(x_0)), d_2^\tau(f(x) - f(x_0)), \ldots, d_m^\tau(f(x) - f(x_0)), 0). \tag{2.29}$$

Conversely, if $x_0 \in Q$ *satisfies condition* (2.28) *with* $\hat{F}$ *being defined by* (2.29) *for some* $M > 0$ and $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n) \in \mathbb{R}^m_+$ *with* $\alpha^i > 0,\ i \in I$, *then* x_0 *is a Hartley properly efficient point of* (P).

Proof. Use Theorem 2.5 and observe from Lemma 2.2 that $\hat{f}(x) \geqslant 0 \Leftrightarrow \hat{F}(x) \geqslant 0,\ x \in Q$. □

REMARK 2.4. In [14] Huang and Yang consider Problem (P) where $m = s$ and d_i is the ith unit vector of $\mathbb{R}^m$, $i = 1, 2, \ldots, m$. In other words, they assume that $D = \mathbb{R}^m_+$ and consider the Geoffrion proper efficiency of x_0. For $\alpha^i > 0,\ i = 1, 2, \ldots, m$, they introduce the following function depending on a parameter $r \in \mathbb{R}$:

$$\bar{p}(x) = \sum_{i\in I} \alpha^i f^i(x) + r\max(f^1(x) - f^1(x_0), f^2(x) - f^2(x_0), \ldots, f^m(x) - f^m(x_0), 0)$$

and prove in [14, Theorem 3.1] that a R^m_+-efficient point x_0 is a Geoffrion properly efficient point of (P) if and only if there exists $r > 0$ such that x_0 is a minimizer of $\bar{p}(\cdot)$ on Q. By taking $d_i,\ i \in I$, to be the ith unit vector of $\mathbb{R}^m$ we see that this result is a consequence of Corollary 2.7. From Corollary 2.7 it follows that in the formulation of Theorem 3.1 of [14] the assumption that x_0 is a R^m_+-efficient point is superfluous.

Now let us introduce the function

$$t'(x, x_0) = \min\left\{\sum_{i\in I} \alpha^i d_i^\tau (f(x) - f(x_0)) : \alpha^i \geqslant 0,\ i \in I,\ \sum_{i\in I} \alpha^i = 1\right\}.$$

THEOREM 2.6.

1. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *if and only if there exist* $M > 0$ *and* $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m_+$ *with* $\alpha^i > 0,\ i \in I$, *such that the function* (2.25) *satisfies condition* (2.26).
2. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *if and only if there exists* $M > 0$ *such that*

$$\min_{x \in Q} \hat{\mathbb{F}}(x) = \hat{\mathbb{F}}(x_0) = 0, \tag{2.30}$$

where $\hat{\mathbb{F}}$ *is defined by*

$$\hat{\mathbb{F}}(x) = \hat{\mathbb{F}}(M, x) = t'(x, x_0) + M \max_{j \in I} d_j^\tau (f(x) - f(x_0)). \tag{2.31}$$

3. *A point* $x_0 \in Q$ *is a Hartley properly efficient point of* (P) *if and only if there exists* $M > 0$ *such that* (2.30) *is satisfied where, instead of* (2.31), $\hat{\mathbb{F}}$ *is defined by*

$$\hat{\mathbb{F}}(x) = \hat{\mathbb{F}}(M,x) = t'(x,x_0) + M \max(d_j^\tau (f(x) - f(x_0)), \ldots, d_m^\tau (f(x) - f(x_0)), 0).$$

Proof. The first part of Theorem 2.6 is a consequence of Theorem 2.5. The second and third parts can be established by the same argument used in the proof of Theorem 2.2. □

COROLLARY 2.8. *Let* x_0 *be a D-efficient point of* (P) *and* f *be not constant on* Q. *Then* $Q'(x_0) \neq \emptyset$, *and* x_0 *is a Hartley properly efficient point of* (P) *if and only if*

$$\sup_{x \in Q'(x_0)} \frac{-t'(x, x_0)}{\max_{j \in I} d_j^\tau (f(x) - f(x_0))} < +\infty.$$

The proof is similar to that of Corollary 2.1 and is omitted.

3. Necessary Conditions for Hartley Proper Efficiency in Nonsmooth Vector Optimization Problems

We first recall that the ordering cone D for (P) is the polyhedral cone defined in (2.15). Now we introduce some notions of Nonsmooth Analysis [7]. Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke subdifferential of Ψ at $x_0 \in \mathbb{R}^n$ is the set

$$\partial \Psi(x_0) = \{\xi \in \mathbb{R}^n : x^\tau \xi \leqslant \Psi^0(x_0; x), \ \forall x \in \mathbb{R}^n\},$$

where

$$\Psi^0(x_0; x) = \limsup_{\substack{x' \to x_0 \\ t \downarrow 0}} \frac{1}{t} [\Psi(x' + tx) - \Psi(x')].$$

The Clarke tangent cone and the Clarke normal cone of a subset $C \subset \mathbb{R}^n$ at $x_0 \in C$ are denoted by $T_C(x_0)$ and $N_C(x_0)$, respectively. Recall that

$$T_C(x_0) = \{\eta \in \mathbb{R}^n : \rho_C^0(x_0; \eta) = 0\},$$

$$N_C(x_0) = \{\xi \in \mathbb{R}^n : \xi^T \eta \leqslant 0, \ \forall \eta \in T_C(x_0)\},$$

where $\rho_C(x) = \rho(x, C)$ i.e., $\rho_C(x)$ is the distance from $x \in \mathbb{R}^n$ to C.

From now on we assume that

$$Q = \{x \in C : g^j(x) \leqslant 0,\ j = 1, 2, \ldots, p,\ h^l(x) = 0,\ l = 1, 2, \ldots, q\}$$

where $C \subset \mathbb{R}^n$ is a closed subset, and g^j and h^l are given functions. We also assume that all functions f^i, g^j and h^l are locally Lipschitz. Let $x_0 \in Q$ and let

$$J(x_0) = \{j : g^j(x_0) = 0\}.$$

We say that condition (CQ) holds at x_0 if there do not exist $\mu^j \geqslant 0,\ j \in J(x_0)$, and $r^l \in \mathbb{R},\ l = 1, 2, \ldots, q$, such that $\sum_{j \in J(x_0)} \mu^j + \sum_{l=1}^{q} |r^l| \neq 0$ and

$$0 \in \sum_{j \in J(x_0)} \mu^j \partial g^j(x_0) + \sum_{l=1}^{q} r^l \partial h^l(x_0) + N_C(x_0),$$

where $\partial g^j(x_0)$ and $\partial h^l(x_0)$ are the Clarke subdifferentials of g^j and h^l at x_0, and $N_C(x_0)$ stands for the Clarke normal cone to C at x_0. We will denote by $\partial(d_i^\tau f)(x_0)$ the Clarke subdifferential of $d_i^\tau f(\cdot)$ at x_0.

As an application of Theorem 2.5 of the previous section we obtain the following result.

THEOREM 3.1. *Let $x_0 \in Q$ and let condition* (CQ) *hold. If $x_0 \in Q$ is a Hartley properly efficient point of (P) then there exist $\lambda^i > 0,\ i = 1, \ldots, m$, $\mu^j \geqslant 0,\ j \in J(x_0),\ r^l \in \mathbb{R}, l = 1, 2, \ldots, q$, such that*

$$0 \in \sum_{i=1}^{m} \lambda^i \partial(d_i^\tau f)(x_0) + \sum_{j \in J(x_0)} \mu^j \partial g^j(x_0) + \sum_{l=1}^{q} r^l \partial h^l(x_0) + N_C(x_0). \tag{3.1}$$

Proof. Setting $\alpha^i = 1,\ i \in I$, and applying Theorem 2.5 we see that x_0 is a solution of the problem of minimizing the function $\hat{f}(x)$ subject to $x \in Q$ where

$$\hat{f}(x) = \sum_{i=1}^{m} d_i^\tau(f(x) - f(x_0)) + mM \max_{j \in I}\ d_j^\tau(f(x) - f(x_0))$$

and M is a Hartley constant of (P) at x_0. By a result of Clarke [7, Theorem 6.1.1], we must find $\beta \geqslant 0,\ \mu^j \geqslant 0, j \in J(x_0),\ r^l \in \mathbb{R},\ l = 1, 2, \ldots, q$, such that

$$\beta + \sum_{j \in J(x_0)} \mu^j + \sum_{l=1}^{q} |r^l| \neq 0$$

and

$$0 \in \beta \partial \hat{f}(x_0) + \sum_{j \in J(x_0)} \mu^j \partial g^j(x_0) + \sum_{l=1}^{q} r^l \partial h^l(x_0) + N_C(x_0). \tag{3.2}$$

By condition (CQ) $\beta \neq 0$ and hence, we can set $\beta = 1$. Now, making use of [7, Proposition 2.3.12] and [7, Proposition 2.3.3] we have

$$\partial\hat{f}(x_0) \subset \sum_{i=1}^{m} \partial(d_i^\tau f)(x_0) + mM \operatorname{co}\left(\bigcup_{j\in I} \partial(d_j^\tau f)(x_0)\right). \tag{3.3}$$

From (3.3) it follows that

$$\partial\widehat{f}(x_0) \subset \sum_{i=1}^{m} (1 + \delta^i mM)\partial(d_i^\tau f)(x_0)$$

for some $\delta^i \geqslant 0$, $i = 1, 2, \ldots, m$, with $\sum_{i=1}^m \delta^i = 1$. Combining this with (3.2) and recalling that $\beta = 1$ we obtain (3.1) where $\lambda^i = 1 + \delta^i mM > 0$, $i = 1, 2, \ldots, m$. □

To give conditions under which necessary conditions given in Theorem 3.1 become sufficient ones for Hartley proper efficiency, we need some new generalized convexity notions. Section 4 is devoted to introducing and characterizing these notions.

4. Near Invexity and Near Infineness

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz vector-valued map with components φ^i, $i \in I := \{1, 2, \ldots, m\}$:

$$\varphi(x) = (\varphi^1(x), \varphi^2(x), \ldots, \varphi^m(x))^\tau, \quad x \in \mathbb{R}^n.$$

Let C be a subset of $\mathbb{R}^n$ and let $x_0 \in C$.

The following generalized convexity notion is taken from [19].

DEFINITION 4.1. A vector-valued map φ is called invex on C at $x_0 \in C$ if

$$\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I, \quad \exists \eta \in T_C(x_0),\ \text{such that}$$
$$\varphi^i(x) - \varphi^i(x_0) \geqslant \xi_i^\tau \eta, \quad i \in I.$$

When $m = 1$, φ is of class C^1, and $C = \mathbb{R}^n$ this definition reduces to a generalized convexity notion first given by Hanson [12]. The reader is referred to [19] for characterizations of invexity in the sense of Definition 4.1 and for applications of this property to nonsmooth alternative theorems and optimization theory.

DEFINITION 4.2. A vector-valued map φ is called nearly invex on C at $x_0 \in C$ if

$$\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I, \quad \exists \eta_k \in T_C(x_0),\ k = 1, 2, \ldots,\ \text{such that}$$
$$\varphi^i(x) - \varphi^i(x_0) \geqslant \limsup_k \xi_i^\tau \eta_k, \quad i \in I. \tag{4.1}$$

Obviously, invexity $\Longrightarrow$ near invexity. An example will be given later to prove that the converse implication is no longer true. In other words, the class of nearly invex maps is strictly broader than that of invex maps introduced in [19].

Before giving characterizations of nearly invex maps let us introduce some notation. If $\xi_i \in \mathbb{R}^n$, $i \in I$, then we denote by $\boldsymbol{\xi}$ the $n \times m$ – matrix with columns ξ_i and we write $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)$. Thus, if $x \in \mathbb{R}^n$ then $\boldsymbol{\xi}^\tau x$ is an element of $\mathbb{R}^m$ with components $\xi_i^\tau x$. For $A \subset \mathbb{R}^n$ we write $\boldsymbol{\xi}^\tau A = \bigcup\{\boldsymbol{\xi}^\tau a : a \in A\}$.

THEOREM 4.1. *The following statements are equivalent*:

(a) *φ is nearly invex on C at $x_0 \in C$.*

(b) $\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I,$

$$\varphi(x) - \varphi(x_0) \in \overline{\boldsymbol{\xi}^\tau T_C(x_0) + \mathbb{R}^m_+}. \tag{4.2}$$

(c) $\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I,\ \forall \bar{\xi} \in N_C(x_0),$
$\exists \eta_k \in \mathbb{R}^n,\ k = 1, 2, \ldots,$ *such that*

$$\varphi^i(x) - \varphi^i(x_0) \geqslant \limsup_k \xi_i^\tau \eta_k,\ \ i \in I, \tag{4.3}$$

$$0 \geqslant \limsup_k \bar{\xi}^\tau \eta_k. \tag{4.4}$$

(d) $\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I,\ \forall \bar{\xi} \in N_C(x_0)$

$$\begin{pmatrix} \varphi(x) - \varphi(x_0) \\ 0 \end{pmatrix} \in \overline{\bar{\boldsymbol{\xi}}^\tau \mathbb{R}^n + \mathbb{R}^m_+ \times \mathbb{R}_+}, \tag{4.5}$$

where the left side of (4.5) *is the vector of* $\mathbb{R}^{m+1}$ *with components* $\varphi^i(x) - \varphi^i(x_0),\ i \in I,$ *and* $0 \in \mathbb{R}$, *and* $\bar{\boldsymbol{\xi}} = (\xi_1, \xi_2, \ldots, \xi_m, \bar{\xi})$.

Proof. (a) $\Rightarrow$ (c) Let $\eta_k \in T_C(x_0)$ be the sequence mentioned in Definition 4.2. Since $\bar{\xi}^\tau \eta_k \leqslant 0$ for all $\bar{\xi} \in N_C(x_0)$ and $\eta_k \in T_C(x_0)$ the validity of inequality (4.4) is obvious.

(c) $\Rightarrow$ (b) Assume to the contrary that (4.2) does not hold for some $x \in C$ and $\xi_i \in \partial\varphi^i(x_0)$, $i \in I$. Then, using a separation theorem and noting that the right side of (4.2) is a closed convex cone we can find a vector $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^m)^z \in \mathbb{R}^m$ such that

$$\sum_{i=1}^m \lambda^i(\varphi^i(x) - \varphi^i(x_0)) > 0 \geqslant \sum_{i=1}^m \lambda^i \xi_i^\tau x + \sum_{i=1}^m \lambda^i y^i \tag{4.6}$$

for all $x \in T_C(x_0)$ and $(y^1, y^2, \ldots, y^m)^\tau \in \mathbb{R}^m_+$. From this we can derive by a standard argument that $\lambda^i \leqslant 0$, $i \in I$, and

$$\sum_{i=1}^m \lambda^i \xi_i^\tau x \leqslant 0, \quad x \in T_C(x_0).$$

Since the normal cone is the nonpositive polar cone of the tangent cone we obtain from the last condition that $\bar{\xi} \in N_C(x_0)$ where

$$\bar{\xi} = \sum_{i=1}^{m} \lambda^i \xi_i. \tag{4.7}$$

By condition (c) there exists a sequence $\eta_k \in \mathbb{R}^n$ such that (4.3) and (4.4) hold. Multiplying both sides of (4.3) by λ^i and summing up the obtained inequalities we get

$$\begin{aligned}\sum_{i=1}^{m} \lambda^i(\varphi^i(x) - \varphi^i(x_0)) &\leqslant -\sum_{i=1}^{m} \limsup_k (-\lambda^i)\xi_i^\tau \eta_k \\ &\leqslant -\limsup_k \sum_{i=1}^{m} (-\lambda^i)\xi_i^\tau \eta_k \\ &= \liminf_k \sum_{i=1}^{m} \lambda^i \xi_i^\tau \eta_k \\ &\leqslant \limsup_k \bar{\xi}^\tau \eta_k \\ &\leqslant 0.\end{aligned}$$

(In the above argument we use (4.7) and (4.4)). This contradicts the first inequality in (4.6).

(b) $\Rightarrow$ (a) Given $x \in C$ and $\xi_i \in \partial\varphi^i(x_0)$, $i \in I$, we can find by (4.2) sequences $\eta_k \in T_C(x_0)$ and $y_k = (y_k^1, y_k^2, \ldots, y_k^m)^\tau \in \mathbb{R}_+^m$ such that, for all $i \in I$,

$$\begin{aligned}\varphi_i(x) - \varphi_i(x_0) &= \lim_{k\to\infty} (\xi_i^\tau \eta_k + y_k^i) \\ &\geqslant \limsup_k \xi_i^\tau \eta_k.\end{aligned}$$

Thus, (a) holds.

(c) $\Rightarrow$ (d) Let $x \in C$, $\xi_i \in \partial\varphi^i(x_0)$, $i \in I$, and $\bar{\xi} \in N_C(x_0)$. Assume to the contrary that (4.5) does not hold. Then, using a separation theorem and noting that the right side of (4.5) is a closed convex cone we can find $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^m)^\tau \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$ such that

$$\sum_{i=1}^{m} \lambda^i(\varphi^i(x) - \varphi^i(x_0)) + \mu \cdot 0 > 0 \geqslant \sum_{i=1}^{m} (\lambda^i \xi_i^\tau + \mu\bar{\xi}^\tau)x + \sum_{i=1}^{m} \lambda^i y^i + \mu r \tag{4.8}$$

for all $x \in \mathbb{R}^n$, $(y^1, y^2, \ldots, y^m)^z \in \mathbb{R}_+^m$ and $r \in \mathbb{R}_+$. From this we can derive that $\lambda^i \leqslant 0$, $i \in I$, $\mu \leqslant 0$ and

$$\sum_{i=1}^{m} \lambda^i \xi_i^\tau + \mu\bar{\xi}^\tau = 0. \tag{4.9}$$

Now, in view of condition (c) there exists a sequence $\eta_k \in \mathbb{R}^n$, $k = 1, 2, \ldots$, such that (4.3) and (4.4) hold. Multiplying both sides of (4.3) by λ^i and both sides of (4.4) by μ and summing up the obtained inequalities we get

$$\begin{aligned}\sum_{i=1}^{m} \lambda^i(\varphi^i(x) - \varphi^i(x_0)) &\leqslant -\left(\sum_{i=1}^{m} \limsup_k (-\lambda^i)\xi_i^\tau \eta_k + \limsup_k (-\mu)\bar{\xi}^\tau \eta_k\right) \\ &\leqslant -\limsup_k \left(\sum_{i=1}^{m} (-\lambda^i)\xi_i^\tau \eta_k - \mu\bar{\xi}^\tau \eta_k\right) \\ &= \liminf_k \left(\sum_{i=1}^{m} \lambda^i \xi_i^\tau + \mu\bar{\xi}^\tau\right)\eta_k \\ &= 0 \ \ (\text{by } (4.9)).\end{aligned}$$

This contradicts the first inequality in (4.8).

(d) $\Rightarrow$ (c) Given $x \in C$, $\xi_i \in \partial\varphi^i(x_0)$, $i \in I$ and $\bar{\xi} \in N_C(x_0)$, we can find by (4.5) sequences $\eta_k \in \mathbb{R}^n$, $y_k = (y_k^1, y_k^2, \ldots, y_k^m)^\tau \in \mathbb{R}_+^m$ and $r_k \in \mathbb{R}_+$ such that for all $i \in I$,

$$\varphi^i(x) - \varphi^i(x_0) = \lim_{k\to\infty} (\xi_i^\tau \eta_k + y_k^i), \qquad 0 = \lim_{k\to\infty} (\bar{\xi}^\tau \eta_k + r_k).$$

From these equalities we derive (4.3) and (4.4) since $y_k^i \geqslant 0$ and $r_k \geqslant 0$. $\square$

REMARK 4.1. The characterizations (c) and (d) in Theorem 4.1 show that near invexity can be expressed in terms of the normal cone $N_C(x_0)$. It is worth noticing that in (c) the points η_k are not required to be elements of $T_C(x_0)$.

COROLLARY 4.1. *Definitions* 4.1 *and* 4.2 *are equivalent if at least one of the following conditions is satisfied*:

(i) *For all* $\xi_i \in \partial\varphi^i(x_0)$, $i \in I$, *the set* $\boldsymbol{\xi}^\tau T_C(x_0) + \mathbb{R}_+^m$ *is closed.*
(ii) $x_0 \in \operatorname{int} C$.
(iii) $m = 1$ i.e., φ *is a real-valued function.*

Proof. From Definition 4.1 it is clear that φ is invex on C at $x_0 \in C$ if and only if

$$\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ \ i \in I, \quad \varphi(x) - \varphi(x_0) \in \boldsymbol{\xi}^\tau T_C(x_0) + \mathbb{R}_+^m.$$

Thus in case (i) the conclusion of Corollary 4.1 is derived from the characterization (b) of Theorem 4.1. To conclude the proof of Corollary 4.1 it suffices to show that each of conditions (ii) and (iii) implies (i). Indeed, in case (ii) $T_C(x_0) = \mathbb{R}^n$. Therefore, $\boldsymbol{\xi}^\tau T_C(x_0)(= \boldsymbol{\xi}^\tau \mathbb{R}^n)$ is a subspace of $\mathbb{R}^m$ and hence, $\boldsymbol{\xi}^\tau T_C(x_0) + \mathbb{R}^m_+$ is closed, as required. To see that (iii) $\Rightarrow$ (i) it is enough to remark that each cone in $\mathbb{R}$ containing $0 \in \mathbb{R}$ must be closed, and that $\boldsymbol{\xi}^\tau T_C(x_0) + \mathbb{R}_+$ is such a cone. □

REMARK 4.2. From Corollary 4.1 it is clear that the class of nearly invex maps at $x_0 \in C$ is strictly broader than that of invex maps only if $m \geqslant 2$ and $x_0 \notin \operatorname{int} C$. The following example 4.1 proves that there do exist nearly invex maps which are not invex. This example is constructed on the basis of Example 2.2.8 of [8] which is used in [8] to prove that the image of a closed convex cone via a linear continuous map may not be closed.

EXAMPLE 4.1. Let $C = \{x = (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha \geqslant 0, \beta \geqslant 0, 2\alpha\gamma \geqslant \beta^2\}$ and $\varphi(x) = \varphi(\alpha, \beta, \gamma) = (\varphi^1(\alpha, \beta, \gamma), \varphi^2(\alpha, \beta, \gamma))^\tau = (-\alpha^2 + \beta, \gamma)^\tau$. Let $x_0 = (\alpha_0, \beta_0, \gamma_0)^\tau = (0, 0, 0)^\tau$. Obviously, $T_C(x_0) = C$ and $\partial\varphi^i(x_0) = \{\xi_i\}$, $i = 1, 2$, where $\xi_1 = (0, 1, 0)^\tau$, $\xi_2 = (0, 0, 1)^\tau$. Setting $\boldsymbol{\xi} = (\xi_1, \xi_2)$ we see that

$$\boldsymbol{\xi}^\tau T_C(x_0) = \{(\beta, \gamma) \in \mathbb{R}^2 : \gamma > 0\} \cup \{(0, 0)\}$$

and for all $x \in C$,

$$\varphi(x) - \varphi(x_0) \in \overline{\boldsymbol{\xi}^\tau T_C(x_0)} = \overline{\boldsymbol{\xi}^\tau T_C(x_0) + \mathbb{R}^2_+}.$$

Thus φ is nearly invex on C at $x_0 \in C$ (see Theorem 4.1). But φ is not invex on C at x_0 since

$$\varphi(x) - \varphi(x_0) \notin \boldsymbol{\xi}^\tau T_C(x_0) + \mathbb{R}^2_+,$$

where $x = (1, 0, 0)^\tau$.

It is remarked in [12] that invexity notions are useful for studying sufficient conditions in optimization problems with inequality constraints since the Lagrange multipliers associated to inequality constraints are nonnegative. Unfortunately, the Lagrange multipliers associated to equality constraints may be negative and hence, invexity notions are not suitable for such constraints. In [19] a modified version of invexity is introduced and is proved to be useful for equality constraints. This is the notion of infineness which is recalled in the following definition.

DEFINITION 4.3. [19]. A vector-valued map φ is called infine on C at x_0 if

$$\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I, \quad \exists \eta \in T_C(x_0) \text{ such that}$$

$$\varphi^i(x) - \varphi^i(x_0) = \xi_i^\tau \eta,\ i \in I.$$

We now generalize this notion. The new generalized version is called near infineness.

DEFINITION 4.4. A vector-valued map φ is called nearly infine on C at $x_0 \in C$ if

$$\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I,\ \exists \eta_k \in T_C(x_0),\ k = 1, 2, \ldots,$$

such that

$$\varphi^i(x) - \varphi^i(x_0) = \lim_{k\to\infty} \xi_i^\tau \eta_k,\ \ i \in I. \tag{4.10}$$

Obviously, infineness $\Rightarrow$ near infineness. Example 4.1 proves that the converse implication is no longer true. Thus, the class of nearly infine maps is strictly broader than that of infine maps.

By arguments similar to those used in the proof of Theorem 4.1 we can obtain the following theorem which gives characterizations of nearly infine maps.

THEOREM 4.2. *The following statements are equivalent*:

(a) *φ is nearly infine on C at $x_0 \in C$.*

(b) $\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I,$

$$\varphi(x) - \varphi(x_0) \in \overline{\boldsymbol{\xi}^\tau T_C(x_0)}.$$

(c) $\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I,\ \forall \bar{\xi} \in N_C(x_0),\ \ \exists \eta_k \in \mathbb{R}^n,\ k = 1, 2, \ldots,$ *such that*

$$\varphi^i(x) - \varphi^i(x_0) = \lim_{k\to\infty} \xi_i^\tau \eta_k,\ \ i \in I,$$
$$0 \geqslant \limsup_k \bar{\xi}^\tau \eta_k.$$

(d) $\forall x \in C,\ \forall \xi_i \in \partial\varphi^i(x_0),\ i \in I,\ \forall \bar{\xi} \in N_C(x_0),$

$$\begin{pmatrix} \varphi(x) - \varphi(x_0) \\ 0 \end{pmatrix} \in \overline{\bar{\boldsymbol{\xi}}^\tau \mathbb{R}^n + \{0\} \times \mathbb{R}_+}.$$

Observe that in the right side of the last inclusion 0 denotes the origin of $\mathbb{R}^m$ while in the left side 0 stands for the origin of $\mathbb{R}$.

COROLLARY 4.2. *Definitions* 4.3 *and* 4.4 *are equivalent if at least one of the following conditions is satisfied*:

(i) *For all $\xi_i \in \partial\varphi^i(x_0)$, $i \in I$, the set $\boldsymbol{\xi}^\tau T_C(x_0)$ is closed.*
(ii) $x_0 \in \operatorname{int} C$.
(iii) $m = 1$ i.e., *φ is a real-valued function.*

The relationships between near invexity and near infineness are given in the following theorem.

THEOREM 4.3. (i) *If φ is nearly infine on C at $x_0 \in C$ then φ is nearly invex on C at x_0.*

(ii) *If the vector-valued map $\begin{pmatrix} \varphi \\ -\varphi \end{pmatrix}$ is nearly invex on C at $x_0 \in C$ then φ is nearly infine on C at x_0.*

Proof. The first part of Theorem 4.3 is obvious. To prove the second one let us take $x \in C$ and $\xi_i \in \partial\varphi^i(x_0),\ i \in I$. Since $-\xi_i \in \partial(-\varphi^i)(x_0),\ i \in I$, and since $\begin{pmatrix} \varphi \\ -\varphi \end{pmatrix}$ is nearly invex we must find $\eta_k \in \mathbb{R}^n$, $k = 1, 2 \ldots,$ such that, for all $i \in I$,

$$\varphi^i(x) - \varphi^i(x_0) \geqslant \limsup_k \xi_i^\tau \eta_k,$$

$$(-\varphi^i)(x) - (-\varphi^i)(x_0) \geqslant \limsup_k -\xi_i^\tau \eta_k.$$

From these inequalities it follows that

$$\limsup_k \xi_i^\tau \eta_k \leqslant \varphi^i(x) - \varphi^i(x_0) \leqslant \liminf_k \xi_i^\tau \eta_k.$$

This proves (4.10), as required. □

REMARK 4.3. If φ is nearly infine on C at x_0 then $\begin{pmatrix} \varphi \\ -\varphi \end{pmatrix}$ may not be nearly invex on C at x_0. In other words, the converse of statement (ii) in Theorem 4.3 is no longer true. The following example taken from [19] illustrates this fact: $C = \mathbb{R}$, $x_0 = 0 \in \mathbb{R}$ and

$$\varphi(x) = \begin{cases} \frac{1}{2}x & \text{if } x \geqslant 0, \\ x & \text{if } x < 0. \end{cases}$$

We conclude this section by introducing a notion which is a combination of near invexity and near infineness.

Let f and h be locally Lipschitz vector-valued with components f^i, $i = 1, 2, \ldots, m$, and h^l, $l = 1, 2, \ldots, q$. We say that $\begin{pmatrix} f \\ h \end{pmatrix}$ is nearly invex-infine on C at x_0 if f is nearly invex on C at x_0 and h is nearly infine on C at

x_0, with the same sequence η_k mentioned in each of these definitions. More exactly, $\begin{pmatrix} f \\ h \end{pmatrix}$ is nearly invex-infine on C at x_0 if

$$\begin{aligned} &\forall x \in C, \quad \forall \xi_i \in \partial f^i(x_0), \; i = 1,2,\ldots,m, \quad \forall \xi'_l \in \partial h^l(x_0), \; l = 1,2,\ldots,q, \\ &\exists \eta_k \in T_C(x_0), \; k = 1,2,\ldots, \text{ such that} \\ &f^i(x) - f^i(x_0) \geqslant \limsup_k \xi_i^\tau \eta_k, \quad i = 1,2,\ldots,m, \\ &h^l(x) - h^l(x_0) = \lim_{k\to\infty} \xi'^\tau_l \eta_k, \quad l = 1,2,\ldots,q. \end{aligned}$$

5. Sufficient Conditions for Hartley Proper Efficiency in Nonsmooth Vector Optimization Problems

The following theorem shows that under the near invex-infineness property the converse statement of the conclusion of Theorem 3.1 is true.

THEOREM 5.1. *Let $x_0 \in Q$ and let $J(x_0) = \{j : g^j(x_0) = 0\}$. Let $\tilde{F}$ be the vector-valued map with components $d_i^\tau f$, $i \in I$, and g^j, $j \in J(x_0)$. Let $\begin{pmatrix} \tilde{F} \\ h \end{pmatrix}$ be nearly invex-infine on C at x_0. If there exist $\lambda^i > 0$, $i = 1,2,\ldots,m$, $\mu^j \geqslant 0$, $j \in J(x_0)$, and $r^l \in \mathbb{R}$, $l = 1,2,\ldots,q$, such that (3.1) holds then x_0 is a Hartley properly efficient point of (P).*

Proof. From (3.1) it follows that there exist $\xi_i \in \partial(d_i^\tau f)(x_0)$, $i = 1,2,\ldots,m$, $\tilde{\xi}_j \in \partial g^j(x_0)$, $j \in J(x_0)$, $\tilde{\tilde{\xi}}_l \in \partial h^l(x_0)$, $l = 1,2,\ldots,q$, and $\bar{\xi} \in N_C(x_0)$ such that

$$-\bar{\xi} = \sum_{i=1}^m \lambda^i \xi_i + \sum_{j\in J(x_0)} \mu^j \tilde{\xi}_j + \sum_{l=1}^q r^l \tilde{\tilde{\xi}}_l. \tag{5.1}$$

Let $x \in Q$ be an arbitrary point. By the near invex-infineness property there exist $\eta_k \in T_C(x_0)$, $k = 1,2,\ldots$, such that

$$\begin{aligned} &\lambda^i d_i^\tau (f(x) - f(x_0)) \geqslant \limsup_k \lambda^i \xi_i^\tau \eta_k, \quad i = 1,2,\ldots,m, \\ &\mu^j (g^j(x) - g^j(x_0)) \geqslant \limsup \mu^j \tilde{\xi}_j^\tau \eta_k, \quad j \in J(x_0), \\ &r^l (g^l(x) - g^l(x_0)) = \lim_{k\to\infty} r^l \tilde{\tilde{\xi}}_l \eta_k, \quad l = 1,2,\ldots,q. \end{aligned}$$

Summing up all these inequalities and equalities, and noting that $g^j(x) \leqslant 0$, $g^j(x_0) = 0$, $h^l(x) = h^l(x_0) = 0$ we obtain

$$\sum_{i=1}^m \lambda^i d_i^\tau (f(x) - f(x_0)) \geqslant \limsup_k \left(\sum_{i=1}^m \lambda^i \xi_i + \sum_{j\in J(x_0)} \mu^j \tilde{\xi}_j + \sum_{l=1}^q r^l \tilde{\xi}_l \right)^\tau \eta_k.$$

From this and (5.1) we get

$$\Psi(x) := \sum_{i=1}^{m} \lambda^i d_i^{\tau}(f(x) - f(x_0)) \geqslant \limsup_{k} -\bar{\xi}^{\tau}\eta_k.$$

But the right side of this inequality is nonnegative since $\bar{\xi} \in N_C(x_0)$ and $\eta_k \in T_C(x_0)$. Since x is an arbitrary point of Q this shows that Ψ attains its minimum at x_0. Applying Corollary 2.3 and observing that $\sum_{i=1}^{m} \lambda^i d_i \in D^{+i}$, we claim that x_0 is a Hartley properly efficient point of (P). □

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